# MATHEMATICAL MODEL AND UNIVERSAL ALGORITHM OF THE SOLUTION OF A NONLINEAR HEAT-CONDUCTION PROBLEM ON THE BASIS OF A FINITE-DIFFERENCES METHOD

A. I. Isakeev and A. V. Zaitsev

UDC 536.12:518.61

A method is proposed for the numerical solution of a nonlinear heat conduction problem with anisotropy, shape complexity, internal heat liberation, and other complicating factors taken into account.

The necessity to develop universal programs for the machine design and optimization of evaporative cooling systems of electrotechnical equipment in indubitable. One of the fundamental calculational procedures of such programs is the solution of a heat conduction problem taking account of the real process progress conditions: the dependence of the coefficients of heat transfer, thermophysical, electrophysical and other properties, as well as internal bulk heat liberation on the temperature, anisotropy, shape complexity, and diversity of boundary conditions (BC). Special attention has recently been paid to the development of universal machine-oriented algorithms [1-4]. Application of the energy balance method [5, 6] to construct finite-difference equations for the numerical solution of heat conduction problems permits the creation of a universal algorithm, and the possibility appears for taking account of the advantages of the fundamental competitive method – the method of finite elements [7, 8] – such as utilization of substantially nonregular meshes to partition the body into elements, a large quantity of element types and shapes being used, taking account of the complex inhomogeneous properties of the material, the simplicity of boundary condition representations, and the creation of universal programs. The advantages inherent in the finite difference method remain here: simplicity of problem formulation, initial data preparation, absence of the necessity to compute the stiffness matrix in each step when solving nonlinear problems.

Let us consider the application of the energy balance method to formulate and solve a heat conduction problem with the complicating factors listed above taken into account.

The energy balance can be written in the following form

$$Q_\lambda + Q_V + Q_s = 0 \tag{1}$$

for any body element in the stationary case.

The characteristic components $Q_S$ for the analysis of evaporative systems are the heat transfer to the intermediate heat carrier during boiling or convection and to the surrounding air (BC of the third kind)

$$Q_\alpha = \alpha(T) S_\alpha (T - T_m), \tag{2}$$

heat elimination with a given heat flux density (BC of the second kind)

$$Q_q = q(T) S_q, \tag{3}$$

thermal contact in the solid body (for ideal contact, BC of the fourth kind)

$$Q_R = S_c (T - T_c)/R(T) \tag{4}$$

and the thermal insulation condition $Q_{in} = 0$.

Different BC can be given simultaneously on different element faces:

$$Q_s = Q_\alpha - Q_q + Q_R. \tag{5}$$

Leningrad Technological Refrigeration Industry Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 53, No. 2, pp. 307-313, August, 1987. Original article submitted May 26, 1986.

 0022-0841/87/5302-0970$12.50 

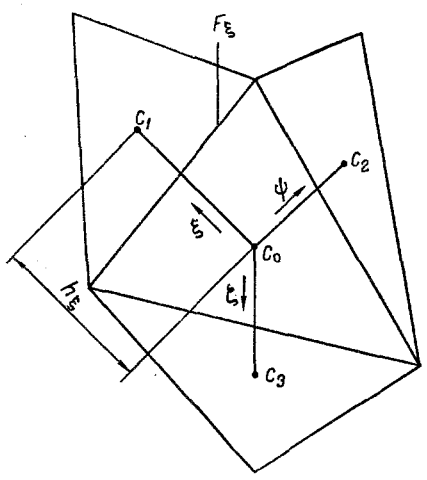


Fig. 1

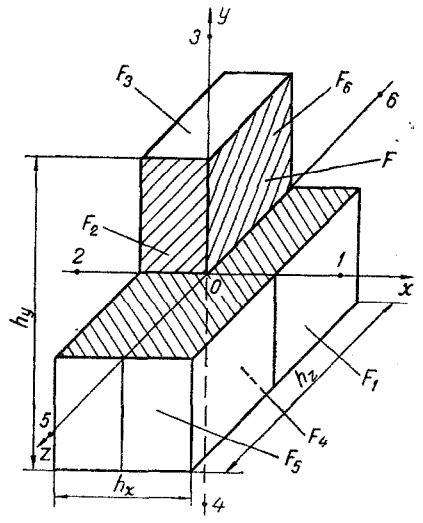


Fig. 2

Fig. 1. Relation between nodal points on a nonregular triangular mesh.

Fig. 2. Surface element of a body in a Cartesian coordinate system.

In the presence of a distributed heat liberation source

$$Q_V = q_V(T)V. \tag{6}$$

We consider determination of $Q_\lambda$ in the general case in an example of a plane triangular element (Fig. 1). The nodal points $C_0$-$C_3$ are the centers of mass of the appropriate triangular elements. The quantity of heat transmitted from element with center at the point $C_1$ to element with center $C_0$ can be determined from the formula

$$Q_\xi = \frac{\lambda_\xi}{h_\xi} F_\xi \sin\varphi (T_1 - T_0) = k_1 (T_1 - T_0),$$

therefore,

$$Q_\lambda = Q_\xi + Q_\psi + Q_\zeta = k_1(T_1 - T_0) + k_2(T_2 - T_0) + k_3(T_3 - T_0). \tag{7}$$

After substituting (7) into (1) and manipulating, we obtain

$$T_0 = \Big[\sum_n k_n T_n + Q_s(T_0) + Q_V(T_0)\Big] \Big/ \sum_n k_n. \tag{8}$$

We later examine a particular case for illustration: an orthotropic body in Cartesian coordinates with spatial mesh steps $h_x$, $h_y$, $h_z$, respectively, in the coordinates x, y, and z.

Equation (8) is converted to the form

$$T_0 = \left[\frac{\lambda_x}{h_x}(F_1T_1 + F_2T_2) + \frac{\lambda_y}{h_y}(F_3T_3 + F_4T_4) + \frac{\lambda_z}{h_z}(F_5T_5 + F_6T_6) + Q_s(T_0) + Q_V(T_0)\right] \Big/ \left[\frac{\lambda_x}{h_x}(F_1 + F_2) + \frac{\lambda_y}{h_y}(F_3 + F_4) + \frac{\lambda_z}{h_z}(F_5 + F_6)\right]. \tag{9}$$

The surfaces $F_1$-$F_6$ can be determined for any nodal point by the formulas

$$F_1 = c_1\frac{h_yh_z}{4}; \quad F_2 = c_2\frac{h_yh_z}{4}; \quad F_3 = c_3\frac{h_xh_z}{4};$$
$$F_4 = c_4\frac{h_xh_z}{4}; \quad F_5 = c_5\frac{h_xh_y}{4}; \quad F_6 = c_6\frac{h_xh_y}{4}, \tag{10}$$

where $c_1$-$c_6$ is the quantity of elementary volumes $h_x h_y h_z/8$, one of whose faces belongs to the surfaces $F_1$-$F_6$, respectively. It follows from Fig. 2 that

$$c_1 + c_2 = c_3 + c_4 = c_5 + c_6 = \sum_{n=1}^{6} c_n/3 = N,$$

where N is the number of elementary volumes comprising the element under consideration with center at the point 0.

The surface of an element lying on the body boundary (shaded surface in Fig. 2) can be determined from the formula

$$S = lh_x h_y/4 + mh_x h_z/4 + ph_y h_z/4. \tag{11}$$

Analysis of all possible variations of the surface point locations shows that the quantities ℓ. m, p can be determined by knowing the number of edges $h_x$, $h_y$, and $h_z$ connecting the surface nodal point under consideration (the center of the element) and the adjacent nodal points lying on the boundary surface. Let $r_x$ denote the number of edges $h_x$, $r_y$ the number of edges $h_y$, and $r_z$ the number of edges $h_z$. Then

$$l = r_x + r_y - r_z, \quad m = r_x - r_y + r_z, \quad p = -r_x + r_y + r_z. \tag{12}$$

For instance, for the element displayed in Fig. 2 we have $r_x = 2$, $r_y = 1$, $r_z = 2$, ℓ = 1, m = 3, p = 1, $S = h_x h_y/4 + 3h_x h_z/4 + h_y h_z/4$.

If the surface nodal point is on the boundary between two or more surfaces with different boundary conditions, then 1/2 is appended for the computation of the numbers $r_x$, $r_y$, and $r_z$ lying on the boundary.

On the basis of (2)-(6), (9)-(11), an algorithm is created that permits automatically obtaining a system of equations of the type (9) to determine the temperature fields in bodies of complex shape in rectangular and cylindrical coordinates. In a program realized on an electronic computer a subprogram is used to analyze whether a nodal point belongs to an external, internal, or surface domain of the body with a specific boundary condition. In the case when λ is independent of the temperature, construction of the coefficients $k_n$ in (8) for 300 nodal points on the ES-1022 computer does not take more than one minute of machine time.

For comparison we say that analogous work performed in conformity with the algorithm [4] on the M-222 computer, which has approximately half the fast-response of the ES-1022, takes up 5 min of machine time.

In the numerical integration of the heat conduction equation it is ordinarily reduced to a system of linear algebraic equations for whose solution an extensively developed mathematical apparatus is used. Computation of the temperature field under conditions similar to the real ones results in a system of nonlinear equations, of the type (8), say, which is solved by iteration methods. In particular, application of the Zeidel method with conversion of the nonlinear terms in all nodal points on each iteration by the results of the preceding iteration [9] is known. The convergence of such a process in application to a system of equations of the type (8) has not been proved.

The condition for approximation of the original differential equation and boundary conditions by a difference scheme when using the energy balance method is satisfied automatically since (8) is obtained from the same relationships as the differential equations with BC in the passage from infinitesimal increments to finite steps. Therefore, satisfaction of the stability condition [6] is necessary for convergence of the iterative process.

The nonlinear terms $Q_S(T_0)$ and $Q_V(T_0)$ are usually a source of instability of the difference scheme (8). In practice, for all real problems it is possible to write

$$Q_s(T_0) + Q_V(T_0) = \sum_i (a_i T_0 + b_i)^{e_i} = f(T_0).$$

Insertion of the error $\delta T_0$ in the determination of $T_0$ in a certain iteration results in change of $f(T_0)$: $f(T_0 + \delta T_0) = f(T_0) + \delta T_0 f'(T_0) + O(\delta T_0^2) \approx f(T_0) + \delta T_0 \sum_i a_i e_i (a_i T_0 + b_i)^{e_i - 1}$ in the next

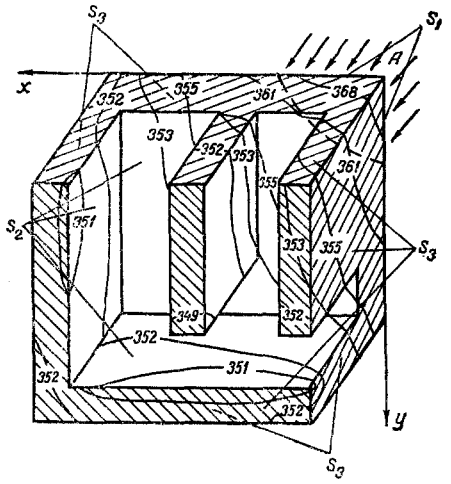


Fig. 3

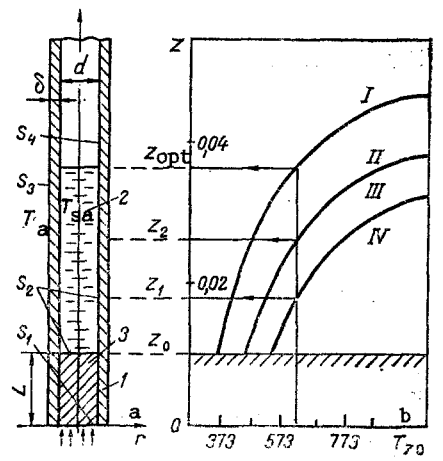


Fig. 4

Fig. 3. Temperature distribution in the evaporator wall of a two-phase siphon.

Fig. 4. Vapor effect in a heat pipe: a) heat pipe construction, b) boiling mode change along the tube length; I) no heat transfer; II) bubbling mode; III) transition; IV) film. $T_{z0}$, °K; z, m.

iteration. Therefore, in order for the calculational errors to have the tendency to decrease from iteration to iteration it is necessary that

$$\left|\frac{\sum_i a_i e_i (a_i T_0 + b_i)^{e_i - 1}}{\sum_n k_n}\right| < 1. \tag{13}$$

Condition (13) is satisfied only for a definite relationship between the thermophysical properties of the body and the characteristics of the heat liberation and heat exchange processes on the body boundaries, on the one hand, and the magnitudes of the difference mesh steps in both the numerator and denominator of (13), on the other. It should be noted that even upon satisfaction of the stability condition there is a serious obstacle to utilization of the difference scheme (8) – too slow convergence, in particular, for BC of the third kind assigned on certain surfaces. It is proposed to introduce a coefficient ω analogous to the relaxation coefficient in relaxation methods

$$T_0^{j+1} = T_0^j + \omega \left| \frac{\sum_n k_n T_n + Q_s(T_0^j) + Q_V(T_0^j)}{\sum_n k_n} - T_0^j \right|$$

to increase the rate of convergence in the difference scheme (8).

The coefficients ω are given separately for each group of nodal points characterizing their kind of nonlinearity. Numerical experiments show that $0 < \omega < 2$. Introduction of the coefficients ω permitted diminution of the computation time in different problems three or more times. Optimal values of ω were determined in numerical experiments by using special test problems.

The most widespread example of solving the nonlinear heat-conduction problem is the hardening or cooling of ingots [10, 11]. Programs permitting the solution of similar problems are presented in [12], in particular, the most effective program for solving the nonlinear one-dimensional nonstationary problem by solving a system of difference equations by an iteration method [12, p. 185]. The Newton method is used to construct the system of difference equations and the solution is by the factorization method at each iteration step. The algorithm proposed on the basis of the finite difference method was realized in the form of a

FORTRAN program with the application of the same principles and error estimates. Appropriate numerical results are obtained. The computation time of the temperature field on an "Élektronika DZ-28" electronic computer was approximately 14 min in each time layer for the algorithm proposed.

Let us examine several examples.

1. Temperature Distribution in a Two-Phase Thermal Siphon Wall. Represented in Fig. 3 is a 1/8 part of a two-phase thermal siphon evaporator to cool a power semiconducting device, formed by three planes of symmetry. The following conditions are given on the body boundaries

$$Qs=\begin{cases}-qS, & S=S_1,\\ \alpha(T_s-T_{sa})S, & S=S_2, \quad \alpha=c(T_s-T_{sa})^nP^m,\\ 0, & S=S_3.\end{cases}$$

The step in all three spatial mesh directions is chosen constant, h = 0.0025 m. The body being computed consists of 1224 elements with volumes from $h^3/8$ (corner elements) to $h^3$ (interior elements).

The temperature field in an evaporator from an AMts alloy ($\lambda$ = 165 W/(m·K)) sec with coolant R113 as heat carrier (c = 1449, n = 0.62, m = 0.36) is presented in Fig. 3 for $T_{ta}$ = 344.3 K, q = $3.26\cdot10^5$ W/m$^2$.

The computed data were compared with experimental results obtained by thermometry on a two-phase thermal siphon model by copper–constantan thermocouples mounted at different points. The greatest discrepancy was observed at the center of the heat delivery surface (the point A in Fig. 3) and was no more than 4%.

2. Investigation of the Vapor Effect in a Heat Pipe. The copper heat pipe 1 (Fig. 4a) to cool a radio electronics element is filled with isopropyl alcohol 2. Heat from the object being cooled is delivered to a monel ($\lambda$ = 25 W/(m·K)) endface of the pipe 3. A vapor effect is observed within the pipe: the existence of film boiling mode is possible near the section $z = z_0$ and then the transition from film to bubble boiling sets in at a certain level $z = z_1$, and the bubble mode at the level $z = z_2$. The heat-transfer coefficient can be determined from the formulas [13]

$$\alpha_1=\begin{cases}21{,}1\,(T_s-T_{sa})^2, & 278<T_s-T_{sa}<300{,}1,\\ 3{,}08\cdot10^8\,(T_s-T_{sa})^{-3}, & 300{,}1\leqslant T_s-T_{sa}<379{,}3,\\ 256, & T_s-T_{sa}\geqslant379{,}3.\end{cases}$$

The problem is solved under the following boundary conditions

$$Qs=\begin{cases}-qS, & S=S_1,\\ \alpha_1(T_s-T_{sa})S, & S=S_2,\\ \alpha_2(T_s-T_a)S, & S=S_3,\\ 0, & S=S_4.\end{cases}$$

The number of steps in the z coordinate is 41 and in the r coordinate 7, and the problem is symmetric in the coordinate $\phi$.

Computations show that independently of the initial temperature field in the pipe a film boiling mode is set on the whole surface $S_2$; strong heating of the element being cooled (the surface $S_1$) to inadmissible temperatures occurs. The increase in the thermal resistance of the endface 3 because of the increase in L or the diminution of the heat conduction results in a reduction in the maximal temperature $T_{z0}$ on the surface $z = z_0$, improvement of heat transfer to $S_2$ on the whole, and consequently, to a reduction in the temperature of the object being cooled. The dependence of the boiling mode transition boundaries and the optimal length of the evaporation zone $z_{opt}$ (the section on which all the heat deliverable to the endface is transmitted by the heat carrier) on $T_{z0}$ for q = $5.1\cdot10^6$ W/m$^2$, $\alpha_2$ = 10 W/(m$^2$·K), $T_{sa}$ = 355 K, $T_a$ = 293 K, L = 0.008 m, d = 0.007 m, $\delta$ = 0.001 m is presented in Fig. 4b.

## NOTATION

C, center of the element; F, surface area of an element face, $m^2$; L, length, m; P, pressure, Pa; Q, thermal flux, W; R, thermal contact resistance, $(m^2 \cdot K)/W$; S, surface area of the body boundary, $m^2$; T, temperature, °K; a, b, c, e, k, m, p, constants; r, x, y, z, $\varphi$, $\psi$, $\zeta$, coordinates; h, step of the spatial mesh, m; q, surface heat flux density, $W/m^2$; $q_V$, bulk heat flux density, $W/m^3$; $\alpha$, heat-transfer coefficient, $W/(m^2 \cdot K)$; $\delta$, thickness, m; $\lambda$, heat-conduction coefficient, $W/(m \cdot K)$; $\omega$, relaxation coefficient. Subscripts: i, number of the boundary condition; j, number of the iteration; n, number of the nodal point neighbor; q, heat supply; s, body surface; V, volume; m, medium; c, contact surface; in, insulation; sa, saturation; a, air.

## LITERATURE CITED


1. A. M. Astapkovich and É. É. Pakhk, Energomashinostroenie, No. 8, 7-9 (1982).
2. A. L. Lubny-Gertysk, Numerical Methods of Solving Thermal Strength Problems in Energetics [in Russian], Moscow (1981), pp. 18-33.
3. Yu. A. Poveshchenko and Yu. P. Popov, Mathematical Models in the Theory of Heat and Mass Transfer. Materials of an International School-Seminar [in Russian], Minsk (1982), pp. 117-122.
4. E. M. Fedorov, Numerical Methods of Solving Thermal Strength Problems in Energetics [in Russian], Moscow (1981), pp. 34-45.
5. A. P. Vanichev, Izv. Akad. Nauk SSSR, Otd. Tekh. Nauk, No. 12, 1767-1774 (1946).
6. A. A. Samarskii, Theory of Difference Schemes [in Russian], Moscow (1983).
7. G. Strang and J. Fix, Theory of Finite Element Methods [Russian translation], Moscow (1977).
8. R. Gallagher, Finite Elements Method. Principles [Russian translation], Moscow (1984).
9. A. I. Isakeev, I. G. Kiselev, V. M. Lyapunov, et al., Analysis of Temperature Fields of Power Plant Nodes [in Russian], Leningrad (1978).
10. N. I. Kobasko and N. M. Fialko, Promyshl. Teplotekh., 4, No. 6, 46-51 (1982).
11. L. A. Kozdoba, Solution of Nonlinear Heat Conduction Problems [in Russian], Kiev (1976).
12. N. M. Belyaev and A. A. Ryadno, Methods of the Theory of Heat Conduction. Ref. Text for Univ. [in Russian], Pt. 2, Moscow (1982).
13. L. I. Roizen and I. N. Dul'kin, Izv. Vyssh. Uchebn. Zaved., Energ., No. 3, 71-77 (1968).